## Divergent Box Integral 3: $I_4^{D=4-2\epsilon}(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0)$

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The result for the box (see figure) in the unphysical region ($s_{12}<0, s_{23}<0, p_2^2<0, p_4^2<0$) is[1]

$$
\begin{aligned}
I_4^{D=4-2\epsilon}(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0) &= \frac{\mu^{2\epsilon}}{(s_{23}s_{12}-p_4^2p_2^2)} \\
\times & \Bigg[\frac{2}{\epsilon^2}\Big((-s_{12})^{-\epsilon}+(-s_{23})^{-\epsilon}-(-p_2^2)^{-\epsilon}-(-p_4^2)^{-\epsilon}\Big) \\
- & 2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{12}}\right)-2\,\mathrm{Li}_2\left(1-\frac{p_2^2}{s_{23}}\right)-2\,\mathrm{Li}_2\left(1-\frac{p_4^2}{s_{12}}\right)-2\,\mathrm{Li}_2\left(1-\frac{p_4^2}{s_{23}}\right) \\
+ & 2\,\mathrm{Li}_2\left(1-\frac{p_2^2p_4^2}{s_{12}s_{23}}\right)-\ln^2\left(\frac{-s_{12}}{-s_{23}}\right)\Bigg]+\mathcal{O}(\epsilon)\,.
\end{aligned}
$$

In ref. [1] the following auxiliary function, (related to the six-dimensional scalar box), has been introduced for the box function with two non-adjacent external lines off-shell,

$$
\begin{aligned}
\mathrm{Ls}_{-1}^{2\mathrm{me}}(s_{12},s_{23};p_2^2,p_4^2) &= -\mathrm{Li}_2\left(1-\tfrac{p_2^2}{s_{12}}\right)-\mathrm{Li}_2\left(1-\tfrac{p_2^2}{s_{23}}\right)-\mathrm{Li}_2\left(1-\tfrac{p_4^2}{s_{12}}\right)-\mathrm{Li}_2\left(1-\tfrac{p_4^2}{s_{23}}\right) \\
&+\mathrm{Li}_2\left(1-\tfrac{p_2^2p_4^2}{s_{12}s_{23}}\right)-\tfrac{1}{2}\ln^2\left(\tfrac{-s_{12}}{-s_{23}}\right)\,. \qquad (1)
\end{aligned}
$$

Thus the result for the box integral may be written as

$$
\begin{aligned}
I_4^{D=4-2\epsilon}(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0) &= \frac{\mu^{2\epsilon}}{(s_{23}s_{12}-p_4^2p_2^2)} \\
\times & \Bigg[\frac{2}{\epsilon^2}\Big((-s_{12})^{-\epsilon}+(-s_{23})^{-\epsilon}-(-p_2^2)^{-\epsilon}-(-p_4^2)^{-\epsilon}\Big)+2\,\mathrm{Ls}_{-1}^{2\mathrm{me}}(s_{12},s_{23};p_2^2,p_4^2)\Bigg]+\mathcal{O}(\epsilon)\,.
\end{aligned}
$$

Note that $\mathrm{Ls}_{-1}^{2\mathrm{me}}(s_{12},s_{23};p_2^2,p_4^2)$ vanishes as $s_{12}+s_{23}-p_2^2-p_4^2\to 0$, (see the file on dilogarithms). Thus the result for the box integral in six dimensions is

$$
I_4^{D=6}(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{\mathrm{Ls}_{-1}^{2\mathrm{me}}(s_{12},s_{23};p_2^2,p_4^2)}{s_{12}+s_{23}-p_2^2-p_4^2}
$$

For $\epsilon, \ln, \mathrm{Li}_2$ etc, see the file on notation.

The analytic continuation of the above expression is quite delicate[2]. Explicit instructions on the continuation are given in ref. [3]. Here we quote an alternative expression from ref. [4] for which the continuation is easier and which contains one fewer dilogarithm.

$$f^{2me} = \frac{s_{12} + s_{23} - p_2^2 - p_4^2}{s_{12}s_{23} - p_2^2 p_4^2}\,,$$

$$\begin{aligned}
& I_4^{D=4-2\epsilon}(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{\mu^{2\epsilon}}{s_{12}\,s_{23} - p_2^2\,p_4^2} \\
& \Big[\frac{2}{\epsilon^2}\Big((-s_{12}-i\varepsilon)^{-\epsilon} + (-s_{23}-i\varepsilon)^{-\epsilon} - (-p_2^2-i\varepsilon)^{-\epsilon} - (-p_4^2-i\varepsilon)^{-\epsilon}\Big) \\
& +2\,\mathrm{Li}_2\Big[1-(s_{12}+\mathrm{i}\varepsilon)\,f^{2me}\Big] \\
& +2\,\mathrm{Li}_2\Big[1-(s_{23}+\mathrm{i}\varepsilon)\,f^{2me}\Big] - 2\,\mathrm{Li}_2\Big[1-(p_2^2+\mathrm{i}\varepsilon)\,f^{2me}\Big] \\
& -2\,\mathrm{Li}_2\Big[1-(p_4^2+\mathrm{i}\varepsilon)\,f^{2me}\Big]\Big] + \mathcal{O}(\epsilon)
\end{aligned}$$

For a discussion of the pole at $s_{23}s_{12} = p_4^2 p_2^2$ in the physical region see ref.[5].

In the unphysical region the pole at $s_{23}s_{12} = p_4^2 p_2^2$ is only apparent because it is canceled by the numerator. In fact, setting $p_2^2 p_4^2 = s_{12}s_{23}(1-r)$ and performing the expansion in $\epsilon$ we find that the integral may be approximated as

$$\begin{aligned}
& I_4^{D=4-2\epsilon}(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0) = \frac{\mu^{2\epsilon}}{s_{23}s_{12}} \\
\times & \Big\{ -\frac{1}{\epsilon}(2+r) + 2 - \frac{r}{2} + (2+r)(\ln(-s_{12}) + \ln(-s_{23}) - \ln(-p_4^2)) \\
+ & \;\; 2\,\mathrm{L}_0\Big(\frac{-\mathrm{p}_4^2}{-\mathrm{s}_{23}}\Big) + 2\,\mathrm{L}_0\Big(\frac{-\mathrm{p}_4^2}{-\mathrm{s}_{12}}\Big) + \mathrm{r}\Big[\mathrm{L}_1\Big(\frac{-\mathrm{p}_4^2}{-\mathrm{s}_{23}}\Big) + \mathrm{L}_1\Big(\frac{-\mathrm{p}_4^2}{-\mathrm{s}_{12}}\Big)\Big]\Big\} + \mathcal{O}(\epsilon,\mathrm{r}^2)\,,
\end{aligned}$$

where $L_0, L_1$ are defined as,

$$\mathrm{L}_0(\mathrm{r}) = \frac{\ln(\mathrm{r})}{1-\mathrm{r}}\,, \qquad \mathrm{L}_1(\mathrm{r}) = \frac{\mathrm{L}_0(\mathrm{r})+1}{1-\mathrm{r}}\,.$$

The functions $L_0$ and $L_1$ have the property that they are finite as their denominators vanish.

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## References

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