Demonstration of basis set of 16 divergent box integrals

We demonstrate that there are only 16 independent box integrals with collinear or soft singularities. All other divergent box integrals can be derived from this set. The integrals are characterized by the number of internal masses which are equal to zero. Y is the so-called modified Cayley matrix.

$$Y_{ij} = \frac{1}{2}\Big[m_i^2 + m_j^2 - (q_{i-1} - q_{j-1})^2\Big]$$

A necessary condition for a soft singularity is that for at least one value of i,

$$Y_{i\,i} = Y_{i\,i+1} = Y_{i\,i-1} = 0\ .$$

A necessary condition for a collinear singularity is that for at least one value of i,

$$Y_{i\,i} = Y_{i+1\,i+1} = Y_{i\,i+1} = 0\ .$$

Thus in order to have a divergence, we must have at least one internal mass equal to zero.

Integrals with no internal masses

There are five of these integrals

1. $I_4^{\{D=4-2\epsilon\}}(0,0,0,0;s_{12},s_{23};0,0,0,0)$
2. $I_4^{\{D=4-2\epsilon\}}(0,0,0,p_4^2;s_{12},s_{23};0,0,0,0)$
3. $I_4^{\{D=4-2\epsilon\}}(0,p_2^2,0,p_4^2;s_{12},s_{23};0,0,0,0)$, ”two opposite offshellness (easy)”
4. $I_4^{\{D=4-2\epsilon\}}(0,0,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0)$, ”two adjacent offshellness (hard)”
5. $I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,0)$

Integrals with one non-zero internal mass

If we have one non-zero internal mass we can take this without loss of generality to be the last one (m_4). In this case the modified Cayley matrix is

$$Y = \begin{pmatrix} 0 & -\frac{1}{2}p_1^2 & -\frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ -\frac{1}{2}p_1^2 & 0 & -\frac{1}{2}p_2^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ -\frac{1}{2}s_{12} & -\frac{1}{2}p_2^2 & 0 & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix}$$

With s_{12}, s_{23} fixed we can apply four conditions to potentially create a soft or collinear divergence, namely $p_1^2 = 0, p_2^2 = 0, p_3^2 = m_4^2, p_4^2 = m_4^2$. However performing the interchange $p_1^2 \leftrightarrow p_2^2, p_3^2 \leftrightarrow p_4^2$, (with m_4 fixed) corresponds to a relabelling of the diagram. In addition setting either $p_3^2 = m_4^2$ without setting $p_2^2 = 0$, or $p_4^2 = m_4^2$ without setting $p_1^2 = 0$ does not lead to a divergence. If we denote application of the four conditions on $p_1^2, p_2^2, p_3^2, p_4^2$ by $ijkl$ we have following fifteen cases.

$$\begin{array}{ll} (6.) & 1234 \\ (7.) & 123 \equiv 124 \\ (8.) & 12 \\ (9.) & 14 \equiv 23 \equiv 134 \equiv 234 \\ (10.) & 1 \equiv 2 \equiv 13 \equiv 24 \\ & 3 \equiv 4 \equiv 34 \equiv \text{finite} \end{array}$$

The first five lines correspond to the integrals 6-10 given below

6. $I_4^{\{D=4-2\epsilon\}}(0, 0, m^2, m^2; s_{12}, s_{23}; 0, 0, 0, m^2)$

7. $I_4^{\{D=4-2\epsilon\}}(0, 0, m^2, p_4^2; s_{12}, s_{23}; 0, 0, 0, m^2)$

8. $I_4^{\{D=4-2\epsilon\}}(0, 0, p_3^2, p_4^2; s_{12}, s_{23}; 0, 0, 0, m^2)$

9. $I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,m^2;s_{12},s_{23};0,0,0,m^2)$

10. $I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,0,m^2)$

Integrals with two adjacent internal masses

Without loss of generality we can take the two non-zero internal masses to be m_3, m_4. In this case the modified Cayley matrix is

$$Y = \left(\begin{array}{cccc} 0 & -\frac{1}{2}p_1^2 & \frac{1}{2}m_3^2-\frac{1}{2}s_{12} & \frac{1}{2}m_4^2-\frac{1}{2}p_4^2 \\ -\frac{1}{2}p_1^2 & 0 & \frac{1}{2}m_3^2-\frac{1}{2}p_2^2 & \frac{1}{2}m_4^2-\frac{1}{2}s_{23} \\ \frac{1}{2}m_3^2-\frac{1}{2}s_{12} & \frac{1}{2}m_3^2-\frac{1}{2}p_2^2 & m_3^2 & \frac{1}{2}m_3^2+\frac{1}{2}m_4^2-\frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2-\frac{1}{2}p_4^2 & \frac{1}{2}m_4^2-\frac{1}{2}s_{23} & \frac{1}{2}m_3^2+\frac{1}{2}m_4^2-\frac{1}{2}p_3^2 & m_4^2 \end{array} \right)$$

A necessary condition to have any divergence is $p_1^2=0$. This gives the integral (13). Applying either $p_2^2=m_3^2$ or $p_4^2=m_4^2$ gives a pair of integrals related by relabelling (12). Applying both $p_2^2=m_3^2$ and $p_4^2=m_4^2$ gives the first integral (11).

11. $I_4^{\{D=4-2\epsilon\}}(0,m_3^2,p_3^2,m_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$

12. $I_4^{\{D=4-2\epsilon\}}(0,m_3^2,p_3^2,p_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$

13. $I_4^{\{D=4-2\epsilon\}}(0,p_2^2,p_3^2,p_4^2;s_{12},s_{23};0,0,m_3^2,m_4^2)$

Integrals with two opposite internal masses

Without loss of generality we can take the two non-zero internal masses to be m_2, m_4. In this case the modified Cayley matrix is

$$Y = \begin{pmatrix} 0 & \frac{1}{2}m_2^2 - \frac{1}{2}p_1^2 & -\frac{1}{2}s_{12} & \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 \\ \frac{1}{2}m_2^2 - \frac{1}{2}p_1^2 & m_2^2 & \frac{1}{2}m_2^2 - \frac{1}{2}p_2^2 & \frac{1}{2}m_2^2 + \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} \\ -\frac{1}{2}s_{12} & \frac{1}{2}m_2^2 - \frac{1}{2}p_2^2 & 0 & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 \\ \frac{1}{2}m_4^2 - \frac{1}{2}p_4^2 & \frac{1}{2}m_2^2 + \frac{1}{2}m_4^2 - \frac{1}{2}s_{23} & \frac{1}{2}m_4^2 - \frac{1}{2}p_3^2 & m_4^2 \end{pmatrix}$$

In this case we can only have a soft divergence. Setting $p_1^2 = m_2^2, p_4^2 = m_4^2$ or $p_2^2 = m_2^2, p_3^2 = m_4^2$ gives two integrals related by relabelling (15). Setting both conditions gives integral (14).

14. $I_4^{\{D=4-2\epsilon\}}(m_2^2, m_2^2, m_4^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)$

15. $I_4^{\{D=4-2\epsilon\}}(p_1^2, m_2^2, m_4^2, p_4^2; s_{12}, s_{23}; 0, m_2^2, 0, m_4^2)$

Integrals with three internal masses

With only one zero mass there can only be a soft singularity. This requires the two adjacent external lines to satisfy the conditions, $p_1^2 = m_2^2, p_4^2 = m_4^2$.

16. $I_4^{\{D=4-2\epsilon\}}(m_2^2, p_2^2, p_3^2, m_4^2; s_{12}, s_{23}; 0, m_2^2, m_3^2, m_4^2)$

Return to general page on boxes

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \end{pmatrix}$$
(1)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}$$
(2)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}$$
(3)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix}$$
(4)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & 0 \end{pmatrix}$$
(5)

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix}$$
(6)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & 0 \\ \times & \times & 0 & \times \end{pmatrix}$$
(7)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & 0 & \times \\ \times & \times & \times & \times \end{pmatrix}$$
(8)

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix}$$
(9)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & \times \\ \times & \times & \times & \times \end{pmatrix}$$
(10)

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}$$
(11)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \\ \times & 0 & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$
(12)

$$\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$
(13)

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & 0 & \times \\ \times & 0 & 0 & 0 \\ 0 & \times & 0 & \times \end{pmatrix}$$
(14)

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & 0 & \times \\ 0 & \times & \times & \times \end{pmatrix}$$
(15)

$$\begin{pmatrix} 0 & 0 & \times & 0 \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix}$$
(16)

Figure 1: Form of the Cayley matrices for the sixteen divergent integrals