Dilogarithms

$$\begin{aligned} \mathrm{Li}_2(x) &= -\int_0^x \frac{dz}{z}\ln(1-z) \\ &\equiv \frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots \text{ when } |x|\le 1 \end{aligned}$$

$$\mathrm{Li}_2(1)=\frac{\pi^2}{6},\quad \mathrm{Li}_2(-1)=-\frac{\pi^2}{12},\quad \mathrm{Li}_2(\frac{1}{2})=\frac{\pi^2}{12}-\frac{1}{2}\ln^2 2$$

$$\mathrm{Li}_2(-x)+\mathrm{Li}_2(-1/x)=-\frac{\pi^2}{6}-\frac{1}{2}\ln^2 x,\quad x>0$$

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1/x)=\frac{\pi^2}{3}-\frac{1}{2}\ln^2 x-i\pi\ln x,\quad x>1$$

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\frac{\pi^2}{6}-\ln x\ln(1-x)$$

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(-\frac{x}{1-x})=-\frac{1}{2}\ln^2(1-x),\quad x<1$$

$$\mathrm{Li}_2(1-x)+\mathrm{Li}_2(1-\frac{1}{x})=-\frac{1}{2}\ln^2(x),\quad x>0$$

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(\frac{x}{x-1})=\frac{\pi^2}{2}-\frac{1}{2}\ln^2(x-1)+i\pi\ln(\frac{x-1}{x^2}),\quad x>1$$

$$\mathrm{Li}_2(1-x)-\mathrm{Li}_2(1/x)=\frac{1}{2}\ln x\ln\left(\frac{x}{(x-1)^2}\right)-\frac{\pi^2}{6},\quad x>1$$

$$\mathrm{Li}_2(\frac{1}{1+x})-\mathrm{Li}_2(-x)=\frac{\pi^2}{6}-\frac{1}{2}\ln(1+x)\ln\left(\frac{1+x}{x^2}\right),\quad x>0$$

$$\begin{aligned} \mathrm{Li}_2(xy) &= \mathrm{Li}_2(x)+\mathrm{Li}_2(y)-\mathrm{Li}_2\left(\frac{y-yx}{1-yx}\right)-\mathrm{Li}_2\left(\frac{x-yx}{1-yx}\right)-\ln\left(\frac{1-y}{1-yx}\right)\ln\left(\frac{1-x}{1-yx}\right) \\ &- \eta(1-y,\frac{1}{1-yx})-\eta(1-x,\frac{1}{1-yx}) \end{aligned}$$

where

$$\ln(xy)=\ln(x)+\ln(y)+\eta(x,y)$$

$$\mathrm{Li}_2(xy) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2\Big(\frac{-y(1-x)}{(1-y)}\Big) - \mathrm{Li}_2\Big(-\frac{x(1-y)}{(1-x)}\Big) - \frac{1}{2}\ln^2\Big(\frac{(1-y)}{(1-x)}\Big) = 0$$

and from ref. [2] for $a = (P^2 + Q^2 - s - t)/(P^2Q^2 - st)$ we have

$$\mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at) =$$
$$\frac{1}{2}\ln^2\Big(\frac{s}{t}\Big) + \mathrm{Li}_2\Big(1-\frac{P^2}{s}\Big) + \mathrm{Li}_2\Big(1-\frac{P^2}{t}\Big) + \mathrm{Li}_2\Big(1-\frac{Q^2}{s}\Big) + \mathrm{Li}_2\Big(1-\frac{Q^2}{t}\Big) - \mathrm{Li}_2\Big(1-\frac{P^2Q^2}{st}\Big)\,.$$

The dilogarithm can be calculated as

$$\mathrm{Li}_2(x) = \sum_{n=0}^{\infty} B_n \frac{z^n}{(n+1)!}, \quad z = -\ln(1-x)$$

where B_n are the Bernoulli numbers

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$
$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510}, \quad B_{18} = \frac{43867}{798},$$

with $B_{2n+1} = 0$ for $n = 1, 2, \ldots$.

References

[1] L. Lewin, Polylogarithms and associated functions, North Holland (1981); Dilogarithms and associated functions, Macdonald (1958).

[2] A. Brandhuber, B. J. Spence and G. Travaglini, Nucl. Phys. B **706**, 150 (2005) [arXiv:hep-th/0407214]