Six-dimensional box integrals

From the Feynman parametrized result for the $D = 4 - 2\epsilon$ integral

$$I_4^{\{D=4-2\epsilon\}}(p_1^2, p_2^2, p_3^2, p_4^2; (p_1+p_2)^2, (p_2+p_3)^2; m_1^2, m_2^2, m_3^2, m_4^2) = \mu^{2\epsilon}\frac{\Gamma(2+\epsilon)}{r_\Gamma}\int_0^1 d^4a_i \, \frac{\delta(1-\sum_i a_i)}{\Big[a_i a_j Y_{ij} - i\varepsilon\Big]^{2+\epsilon}}$$

We can obtain the corresponding result close to six dimensions, by shifting $\epsilon \to \epsilon - 1$ in the integral. (The overall factors are chosen for convenience).

$$I_4^{\{D=6-2\epsilon\}}(p_1^2, p_2^2, p_3^2, p_4^2; (p_1+p_2)^2, (p_2+p_3)^2; m_1^2, m_2^2, m_3^2, m_4^2) = \mu^{2\epsilon}\frac{\Gamma(n-3+\epsilon)}{r_\Gamma}\int_0^1 d^4a_i \, \frac{\delta(1-\sum_i a_i)}{\Big[a_i a_j Y_{ij} - i\varepsilon\Big]^{n-3+\epsilon}}$$

Y is the so-called modified Cayley matrix.

$$Y_{ij} = \frac{1}{2}\Big[m_i^2 + m_j^2 - (q_{i-1} - q_{j-1})^2\Big]$$

Defining c_i

$$c_i = \sum_j Y_{ij}^{-1}, \quad c_0 = \sum_i c_i$$

we find the n-point integral may be expressed as a sum of $(n-1)$-point integrals and an n-point integral close to six dimensions.

$$I_n^{\{D=4-2\epsilon\}} = \frac{1}{2}\left[-\sum_{i=1}^{n} c_i I_n^{(i)} + (n-5+2\epsilon)c_0 I_n^{\{D=6-2\epsilon\}}\right]$$

In this equation $I_n^{(i)}$ is the $D = 4 - 2\epsilon$ dimensional integral obtained from I_n by removing the propagator between legs $i-1$ and i

For the case of the box integrals ($n=4$) the six-dimensional box is finite so we obtain

$$
\begin{aligned}
& I_4^{\{D=4-2\epsilon\}}(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) = -\frac{1}{2}\Bigg[c_0 I_n^{\{D=6\}}(p_1^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23};m_1^2,m_2^2,m_3^2,m_4^2) \\
+ \quad & c_1 I_3^{\{D=4-2\epsilon\}}(s_{23},p_2^2,p_3^2;m_2^2,m_3^2,m_4^2) + c_2 I_3^{\{D=4-2\epsilon\}}(s_{12},p_3^2,p_4^2;m_1^2,m_3^2,m_4^2) \\
+ \quad & c_3 I_3^{\{D=4-2\epsilon\}}(p_1^2,s_{23},p_4^2;m_1^2,m_2^2,m_4^2) + c_4 I_3^{\{D=4-2\epsilon\}}(p_1^2,p_2^2,s_{12};m_1^2,m_2^2,m_3^2)\Bigg]
\end{aligned}
$$

The six dimensional box is finite; if the triangle integrals are known analytically this relation can be verified numerically as a check of the $4-2\epsilon$-dimensional box integrals.

An approach to calculating the general divergent box integral would be to calculate the most general six-dimensional box, whose $D=4-2\epsilon$-dimensional counterpart would have a divergence. The integral needed is $I_4^{\{D=6\}}(m_2^2,p_2^2,p_3^2,p_4^2;s_{12},s_{23},0,m_2^2,m_3^2,m_4^2)$. The Cayley matrix for this integral is

$$
\begin{pmatrix}
0 & 0 & \frac{1}{2}m_3^2-\frac{1}{2}s_{12} & \frac{1}{2}m_4^2-\frac{1}{2}p_4^2 \\
0 & m_2^2 & \frac{1}{2}m_2^2+\frac{1}{2}m_3^2-\frac{1}{2}p_2^2 & \frac{1}{2}m_2^2+\frac{1}{2}m_4^2-\frac{1}{2}s_{23} \\
\frac{1}{2}m_3^2-\frac{1}{2}s_{12} & \frac{1}{2}m_2^2+\frac{1}{2}m_3^2-\frac{1}{2}p_2^2 & m_3^2 & \frac{1}{2}m_3^2+\frac{1}{2}m_4^2-\frac{1}{2}p_3^2 \\
\frac{1}{2}m_4^2-\frac{1}{2}p_4^2 & \frac{1}{2}m_2^2+\frac{1}{2}m_4^2-\frac{1}{2}s_{23} & \frac{1}{2}m_3^2+\frac{1}{2}m_4^2-\frac{1}{2}p_3^2 & m_4^2
\end{pmatrix}
$$

Back to QCDLoop Home page